Natural deduction: Double negation

- If ϕ is true, then $\neg\neg\phi$ is true
- If $\neg\neg\phi$ is true, then ϕ is true

1.	ϕ	premise
2.	$\neg\neg\phi$	$\neg\neg$ i 1

1.	$\neg\neg\phi$	premise
2.	ϕ	$\neg\neg$ e 1

Natural deduction: Implication elimination

- If ψ and $\psi \to \phi$ are true, then ϕ is true

1.	ψ	premise
2.	$\psi \to \phi$	premise
3.	ϕ	$\to$e 1,2

Now prove that $p, p \to q, p \to (q \to r) \vdash r$

Natural deduction: Modus tolens

- If $\neg\phi$ and $\psi \rightarrow \phi$ are true, then $\neg\psi$ is true

1.	$\neg\phi$	premise
2.	$\psi \rightarrow \phi$	premise
3.	$\neg\psi$	MT 1,2

Now prove that $p \rightarrow (q \rightarrow r), p, \neg r \vdash \neg q$

Natural deduction: Implication introduction

- If under the assumption that ϕ is true, also ψ is true, then $\phi \to \psi$

1.	$\psi \to \phi$	premise
2.	$\neg\phi$	assumption
3.	$\neg\psi$	MT 1,2
4.	$\neg\phi \to \neg\psi$	→i 2-3

Box around temporary conclusions

Now prove that $\vdash (q \to r) \to ((\neg q \to \neg p) \to (p \to r))$

$$p \wedge q \to r \vdash p \to (q \to r)$$

Natural deduction: Or-introduction

- If ψ is true, then $\psi \vee \phi$ is true

1.	ϕ	premise
2.	$\psi \vee \phi$	$\vee$i 1

Natural deduction: Or-elimination

- If all of these conditions are true:
  - under the assumption that φ is true, χ is true
  - under the assumption that ψ is true, χ is true
  - formula $\phi \vee \psi$ is true

then χ is true

Natural deduction: Or-elimination

1.	$\phi \to \chi$	premise
2.	$\psi \to \chi$	premise
3.	$\phi \vee \psi$	premise
4.	ϕ	assumption
5.	χ	$\to$e 1,4
6.	ψ	assumption
7.	χ	$\to$e 2,6
8.	χ	$\vee$e 3,4-5, 6-7

assumptions for both cases in the or

Now prove that $q \to r \vdash p \vee q \to p \vee r$

$p \wedge (q \vee r) \vdash (p \wedge q) \vee (p \wedge r)$

Natural deduction: Not-elimination

- If ϕ and $\neg\phi$ are true, then the formula is a contradiction
- One can conclude anything from a contradiction

1.	$\phi \rightarrow \neg\phi$	premise
2.	ϕ	premise
3.	$\neg\phi$	$\rightarrow$e 1,2
4.	$\bot$	$\neg$e 2,3
5.	χ	$\bot$e 4

contradiction found

anything can be concluded from a contradiction

Now prove that $\neg p \vee q \vdash p \rightarrow q$

Natural deduction: Not-introduction

- If the assumption that ϕ is true leads to a contradiction, then $\neg\phi$ is true

1.	$\phi \rightarrow \neg\phi$	premise
2.	ϕ	assumption
3.	$\neg\phi$	$\rightarrow$i 1,2
4.	$\bot$	$\neg$e 2,3
5.	$\neg\phi$	$\neg$i 2-4

Now prove that $p \rightarrow q, p \rightarrow \neg q \vdash \neg p$

Law of the excluded middle

- Try to proof $p \vee \neg p$

Natural deduction: Overview

- We saw rules for
  - And-introduction, and-elimination
  - Or-introduction, or-elimination
  - Not-introduction, not-elimination
  - Implication-introduction, implication-elimination
  - Double negation
  - Modus tolens

  the three latter rules are actually redundant

Natural deduction: "Emulating" modus tolens

1.	$\phi \rightarrow \psi$	premise
2.	$\neg\psi$	premise
3.	ϕ	assumption
4.	ψ	→e 1,3
5.	$\bot$	¬e 2,4
6.	$\neg\phi$	¬i 3-5

Natural deduction: “Emulating” double negation

1.	ϕ	premise
2.	$\neg\phi$	assumption
3.	$\bot$	$\neg$e 1,2
4.	$\neg\neg\phi$	$\neg$i 2-3